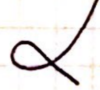
Valuativisations & NMs.

Zariski vs. Etale ~~KS~~ logschemes.

Say $X = \mathbb{A}^2$, $D: y^2 = x^2(x+1)$ 

~~on~~ $X \setminus D$, $M_X = \mathcal{O}^*_{X\setminus D} \cap \mathcal{O}_X$ 'divisorial logstr'.

This formula makes sense on Zariski or étale opens of X:

given $U \xrightarrow{i} X$ étale, (eg. Zariski) $M_X(U) = \mathcal{O}^*_{X\setminus D}(U) \cap \mathcal{O}_X(U)$

$$= \{ f \in \mathcal{O}_X(U) \mid f|_{i^{-1}(X\setminus D)} \text{ invertible} \}$$

But is more interesting if we allow Etale, as then can take cover like ~~$U: (w^2 = x+1)$~~

$$U = \frac{k[x,y,w]}{(w^2-(x+1))} \setminus (x+1=0) \xrightarrow[\text{etale}]{i} X,$$

and i^*D has two irred comps, $y = \pm xw$.

So the stalk of $\overline{M}_X$ at the pt $(x=y=0)$ becomes $\mathbb{N}^2$ (gen. by classes of $y-xw$, $y+xw$).

OTOH, if we insist on doing Zariski logstr, then I'm not certain what happens --- I think stalk is $\mathbb{N}$, gen. by $y^2-x^2(x+1)$.

Later, 'Zariski log str.' will be technically convenient. How to define? Could say 'sheaf of monoids on X_{Zar}', but gets messy.

Better approach (taken from Niziol):

let $\varepsilon: X_{et} \to X_{Zar}$, ~~Given a logscheme (X, M_X)~~,
$(U \to X) \mapsto (\text{image of } U \text{ in } X)$. ~~Say~~ Zariski if

Given $(X, M_{X_{et}})$, can take $\varepsilon_* M_{X_{et}}$,

ie $$\varepsilon_* M_{X_{et}}(\underset{\text{Zariski open}}{U}) = M_{X_{et}}(\underset{\text{view as étale}}{U})$$

Given $(X, M_{X_{Zar}})$, can take $\varepsilon^* M_{X_{Zar}}$, ie

$$\varepsilon^* M_{X_{Zar}}(\underset{\text{ét. map to } X}{U}) = M_{X_{Zar}}\left(\text{image of } U\right).$$

Not too hard to check both ~~gives~~ ε_* & ε^* still give pre l.s., probably l.s.. But what happens when we compose? Do we recover original l.s.?

<u>Def</u> $(X, M_{X_{et}})$ is <u>Zariski</u> if the natural map $M_{X_{et}} \to \varepsilon^* \varepsilon_* M_{X_{et}}$ is an iso.

<u>Ex</u> ε_* gives an $\equiv$ between (Zariski log str) & (log str. on Zar. site).

<u>Ex</u> - The eg. from p(1) is <u>not</u> Zariski.
- if take X, D for D <u>SNCD</u>, then is Zariski.

<u>Thm</u> [Niziol, 5.4]: Let X an fs. log sch. Then $\exists$ a <u>log blowup</u> (see later) $\tilde{X} \to X$ s.t. $\tilde{X}$ is Zariski.

often can't be empty...

Log LRS as in notes (~~left~~ adj't:

$$\mathrm{Hom}(Y^{\log}, X) = \mathrm{Hom}(Y, \mathrm{forget}(X)),$$

$X \in \mathrm{LogStr}$
$Y \in \mathrm{PreLogStr}.$

- every Zariski log Sch is nat. a log LRS.

- not every log LRS is a log Sch (eg. take a ~~log~~ LRS that is not a scheme, put triv. log str).

eg. $(\mathbb{R}, C^{\infty})$, $(\mathbb{C}, \mathrm{hol})$ — bcs not every pt has nbhd of cpt opens

$(\mathrm{pt}, \text{local, non Artin ring})$ — if it were, it would be bigger!

Schematic locus

Let X a LRS. 'Being a scheme' is a property of LRS.

Ex: Let $U, V \subseteq X$ be schemes. Then $U \cup V$ is a scheme.

Def: X^{sch} = largest open which is a scheme, makes sense by the above.

~~Also makes sense for~~
If X a Log LRS, then X^{sch} is nat. a log sch.

Valuativity, as in pdf.